# Stats 369. Homework 5.

## Exercise 2.1.

The probability measure of $k$-spin glass model on $\mathbb{Z}_2^n$ gives

$$\mu_{\beta,n}(\boldsymbol{\sigma}) = \frac{1}{Z_n(\beta)} \exp\left\{ \frac{\beta}{\sqrt{2n^{k-2}(k!)}} \langle \boldsymbol{W}, \boldsymbol{\sigma}^{\otimes k} \rangle \right\}. \tag{1}$$

We have

$$\begin{aligned}
\mathbb{E}[Z_n(\beta)^r] &= \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^r} \mathbb{E}\left[ \exp\left\{ \frac{\beta}{\sqrt{2n^{k-2}(k!)}} \langle \boldsymbol{W}, \sum_{a=1}^r (\boldsymbol{\sigma}^a)^{\otimes k} \rangle \right\} \right] \\
&= \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^r} \mathbb{E}\left[ \exp\left\{ \frac{\beta}{\sqrt{2n^{k-1}}} \langle \boldsymbol{G}, \sum_{a=1}^r (\boldsymbol{\sigma}^a)^{\otimes k} \rangle \right\} \right] \\
&= \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^r} \exp\left\{ \frac{\beta^2}{4n^{k-1}} \left\| \sum_{a=1}^r (\boldsymbol{\sigma}^a)^{\otimes k} \right\|_F^2 \right\} \\
&= \int \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^r} \exp\left\{ \frac{\beta^2}{4n^{k-1}} \left\| \sum_{a=1}^r (\boldsymbol{\sigma}^a)^{\otimes k} \right\|_F^2 \right\} \prod_{1\le a<b\le r} \delta(Q_{ab} - \frac{1}{n}\langle \boldsymbol{\sigma}^a, \boldsymbol{\sigma}^b \rangle) \prod_{1\le a<b\le r} \mathrm{d}Q_{ab} \\
&= \int \prod_{1\le a<b\le r} \mathrm{d}Q_{ab} \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^r} \exp\left\{ \frac{n\beta^2}{4}(r + 2\sum_{1\le a<b\le r} Q_{ab}^k) \right\} \\
&\quad \times \prod_{1\le a<b\le r} \frac{n}{2\pi} \int \exp(-i\lambda_{ab}(nQ_{ab} - \langle \boldsymbol{\sigma}^a, \boldsymbol{\sigma}^b \rangle))\mathrm{d}\lambda_{ab} \\
&\doteq \int \prod_{1\le a<b\le r} (\mathrm{d}Q_{ab}\mathrm{d}\lambda_{ab}) \exp\left\{ n \left( \frac{\beta^2 r}{4} + \sum_{a<b} (\frac{\beta^2}{2} Q_{ab}^k - i\lambda_{ab}Q_{ab}) + \log\left( \sum_{\{\sigma^a\}} \exp(i \sum_{1\le a<b\le r} \lambda_{ab}\sigma^a\sigma^b) \right) \right) \right\} \\
&= \int \prod_{1\le a<b\le r} (\mathrm{d}Q_{ab}\mathrm{d}\lambda_{ab}) \exp\left\{ n \cdot S(Q,\lambda) \right\}
\end{aligned}$$

where

$$S_r(Q,\lambda) = \frac{\beta^2 r}{4} + \sum_{a<b} (\frac{\beta^2}{2} Q_{ab}^k - i\lambda_{ab}Q_{ab}) + \log\left( \sum_{\{\sigma^a\}} \exp(i \sum_{1\le a<b\le r} \lambda_{ab}\sigma^a\sigma^b) \right).$$

Substitute $w_{ab} = i\lambda_{ab}$, we have

$$S_r(Q,w) = \frac{\beta^2 r}{4} + \sum_{a<b} (\frac{\beta^2}{2} Q_{ab}^k - w_{ab}Q_{ab}) + \log\left( \sum_{\{\sigma^a\}} \exp(\sum_{1\le a<b\le r} w_{ab}\sigma^a\sigma^b) \right).$$

Taking derivative with respect to $w_{ab}$ and $Q_{ab}$, at the saddle point, it gives

$$Q_{ab} = \langle \sigma^a\sigma^b \rangle_n, \quad w_{ab} = \frac{k\beta^2}{2} Q_{ab}^{k-1},$$

where

$$\mu_n(\sigma^1, \ldots, \sigma^r) = \frac{1}{Z_n} \exp(\sum_{a<b} w_{ab}\sigma^a\sigma^b).$$

At the replica symmetric solution, we have

$$w = \frac{k\beta^2}{2} q^{k-1},$$

and

$$\begin{aligned}
\log Z_n(w) &= \log(\sum_{\{\sigma^a\}} \exp(w \sum_{a<b} \sigma^a\sigma^b)) \\
&= \log(\sum_{\{\sigma^a\}} \exp(\frac{1}{2} w (\sum_{a=1}^{r} \sigma^a)^2)) - \frac{1}{2} wr \\
&= \log(\sum_{\{\sigma^a\}} \mathbb{E}_Z[\exp(Z\sqrt{w}(\sum_{a=1}^{r} \sigma^a))]) - \frac{1}{2} wr \\
&= \log(\mathbb{E}_Z[\sum_{\{\sigma^a\}} \exp(Z\sqrt{w} \sum_{a=1}^{r} \sigma^a))]) - \frac{1}{2} wr \\
&= \log(\mathbb{E}_Z[2\cosh(Z\sqrt{w})^r]) - \frac{1}{2} wr.
\end{aligned}$$

and we have

$$q = \frac{2}{r(r-1)} \partial_w \log Z_n(w).$$

As $r \to 0$, we have

$$q = \mathbb{E}_Z[\tanh^2(Z\sqrt{w})].$$

We have

$$S_r^{\mathrm{RS}}(q, w) = \frac{\beta^2 r}{4} + \frac{\beta^2 r(r-1)}{4} q^k - \frac{r(r-1)}{2} wq + \log(\mathbb{E}_Z[2\cosh(Z\sqrt{w})^r]) - \frac{1}{2} wr.$$

Divide by $r$ and let $r \to 0$, we have

$$\Phi^{\mathrm{RS}}(q) = \frac{\beta^2}{4} + \frac{(k-1)\beta^2}{4} q^k - \frac{k\beta^2}{4} q^{k-1} + \mathbb{E}_Z[\log(2\cosh(\beta\sqrt{kq^{k-1}/2}Z))].$$

Thus, we have

$$\phi^{\mathrm{RS}} = \frac{\beta^2}{4} + \frac{(k-1)\beta^2}{4} q_*^k - \frac{k\beta^2}{4} q_*^{k-1} + \mathbb{E}_Z[\log(2\cosh(\beta Z\sqrt{kq_*^{k-1}/2}))],$$

where

$$q_* = \mathbb{E}_Z[\tanh^2(\beta Z\sqrt{kq_*^{k-1}/2})].$$

## Exercise 2.1. another approach

Since

$$\begin{aligned}\mathbb{E}\left[Z_n(\beta)^r\right] &= 2^{nr}\mathbb{E}_{\sigma^1,\cdots,\sigma^r}\left[\exp\left\{n\left(\frac{\beta^2 r}{4}+\frac{\beta^2}{2}\sum_{1\le a<b\le r}Q_{ab}^k\right)\right\}\right]\\ &= \mathbb{E}_{\sigma^1,\cdots,\sigma^r}\left[\exp\left\{n\left(\frac{\beta^2 r}{4}+\frac{\beta^2}{2}\sum_{1\le a<b\le r}Q_{ab}^k + r\log 2\right)\right\}\right]\\ &= \mathbb{E}_{Q_n}\left[e^{nf(Q_n)}\right], \end{aligned} \tag{2}$$

where $(Q_n)_{ab} = \langle \sigma^a, \sigma^b\rangle/n$, $Q_n \in [-1,1]^{r(r-1)/2}$. To apply Varadhan's lemma, we consider

$$A(\lambda, u) = f(u) - \langle \lambda, u\rangle + \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[e^{n\langle\lambda, Q_n\rangle}\right], \tag{3}$$

Consider the replica symmetry solution $u = q\cdot\mathbf{1}$ and the stationary condition for $\lambda$,

$$\lambda = \frac{k\beta^2}{2}q^{k-1}\cdot\mathbf{1} =: w\cdot\mathbf{1}\,. \tag{4}$$

It holds that

$$\begin{aligned}&\frac{1}{n}\log\mathbb{E}\left[e^{n\langle\lambda, Q_n\rangle}\right] + r\log 2\\ &= \frac{1}{n}\log\left(2^{nr}\cdot\mathbb{E}\left[\exp\left(w\sum_{1\le a<b\le r}\langle\sigma^a,\sigma^b\rangle\right)\right]\right)\\ &= \frac{1}{n}\log\left(\sum_{\sigma^1,\cdots,\sigma^r\in\{\pm1\}^n}\exp\left(\frac{1}{2}w\left\|\sum_{1\le a\le r}\sigma^a\right\|_2^2 - \frac{1}{2}nwr\right)\right)\\ &= \log\left(\sum_{\sigma_1,\cdots,\sigma_r\in\{\pm1\}}\exp\left(\frac{1}{2}\left(\sum_{1\le a\le r}\sigma_a\right)^2\right)\right) - \frac{1}{2}wr\\ &= \log\left(\mathbb{E}_Z\left[2\cosh(Z\sqrt{w})^r\right]\right) - \frac{1}{2}wr\,. \end{aligned} \tag{5}$$